

KAEHLER STRUCTURES ON $K_{\mathbf{C}}/(P, P)$

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Abstract

Let K be a compact connected semi-simple Lie group, let $G = K_{\mathbf{C}}$, and let $G = KAN$ be an Iwasawa decomposition. Given a K -invariant Kaehler structure ω on G/N , there corresponds a pre-quantum line bundle \mathbf{L} on G/N . Following a suggestion of A.S. Schwarz, in a joint work with V. Guillemin, we study its holomorphic sections $\mathcal{O}(\mathbf{L})$ as a K -representation space. We define a K -invariant L^2 -structure on $\mathcal{O}(\mathbf{L})$, and let $H_{\omega} \subset \mathcal{O}(\mathbf{L})$ denote the space of square-integrable holomorphic sections. Then H_{ω} is a unitary K -representation space, but we find that not all unitary irreducible K -representations occur as subrepresentations of H_{ω} . This paper serves as a continuation of that work, by generalizing the space considered. Instead of working with $G/N = G/(B, B)$, where B is a Borel subgroup containing N , we consider $G/(P, P)$, for all parabolic subgroups P containing B . We carry out similar construction, and recover in H_{ω} the unitary irreducible K -representations previously missing. As a result, we use these holomorphic sections to construct a model for K : a unitary K -representation in which every irreducible K -representation occurs with multiplicity one.

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1 INTRODUCTION

Let K be a compact connected semi-simple Lie group, let $G = K_{\mathbf{C}}$ be its com-

plexification, and let $G = KAN$ be an Iwasawa decomposition. Since G and N are complex Lie groups, G/N is a complex manifold, and G acts on G/N by left action. Let T be the centralizer of A in K , so that $H = TA$ is a Cartan subgroup of G . Since H normalizes N , there is a right action of H on G/N . We shall often be interested in the maximal compact group action of $K \times T$. We let $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \mathfrak{t}, \mathfrak{a}, \mathfrak{n}$ denote the Lie algebras of G, K, H, T, A, N respectively.

The following scheme of geometric quantization was suggested by A.S. Schwarz [12]: Equip G/N with a suitable K -invariant Kaehler structure ω , and consider the pre-quantum line bundle \mathbf{L} associated to ω ([5], [11]). The Chern class of \mathbf{L} is $[\omega]$, and \mathbf{L} comes with a connection ∇ whose curvature is ω , as well as an invariant Hermitian structure \langle, \rangle . We denote by $\mathcal{O}(\mathbf{L})$ the space of holomorphic sections on \mathbf{L} . The K -action on G/N lifts to a K -representation on $\mathcal{O}(\mathbf{L})$. Let μ be the $K \times A$ -invariant measure on G/N , which is unique up to non-zero constant. Given a holomorphic section s of \mathbf{L} , we consider the integral

$$\int_{G/N} \langle s, s \rangle \mu .$$

Let $H_\omega \subset \mathcal{O}(\mathbf{L})$ denote the holomorphic sections in which this integral converges. Since μ is K -invariant, H_ω becomes a unitary K -representation space. It was hoped in [12] that every irreducible K -representation occurs with multiplicity one in H_ω (called a *model* by I.M. Gelfand [7]).

By the method of highest weight, the irreducible K -representations can be labeled by the dominant integral weights in \mathfrak{t}^* , up to isomorphism. In a joint work with V. Guillemin [4], we carry out this construction, but find that no matter how ω is chosen, the irreducibles whose highest weights lie on the wall of the Weyl chamber do not occur in the Hilbert space H_ω . Therefore, not all unitary K -irreducibles occur in H_ω . The present paper follows a suggestion of V. Guillemin ([4] p.192), by modifying the space G/N to more general classes of homogeneous spaces. As a result, we manage to recover the unitary K -irreducibles previously missing.

Let $B = HN$ be the Borel subgroup of G . Observe that $(B, B) = N$, hence

$G/N = G/(B, B)$. With this in mind, we can generalize the class of homogeneous spaces considered to $G/(P, P)$, for P a parabolic subgroup of G containing B . Since P is a complex Lie group, so is (P, P) ; hence $G/(P, P)$ is a complex manifold. Clearly G acts on $G/(P, P)$ on the left, and we shall see that a complex subgroup of H normalizes (P, P) , and hence acts on $G/(P, P)$ on the right.

Let $W \subset \mathfrak{t}^*$ denote the open Weyl chamber, and \overline{W} its closure. We say that $\sigma \subset \overline{W}$ is a *cell* if there exists a subset S of the positive simple roots Δ such that

$$(1.1) \quad \sigma = \{x \in \overline{W} ; (x, S) = 0, (x, \Delta \setminus S) > 0\},$$

where the pairing used is the Killing form. This way, \overline{W} is a disjoint union of the cells of various dimensions. Using the Killing form and the almost complex structure, it is convenient to regard the cell σ as contained in any of the spaces $\mathfrak{h}, \mathfrak{t}, \mathfrak{a}, \mathfrak{h}^*, \mathfrak{t}^*, \mathfrak{a}^*$, depending on the context. The cell σ defines a subalgebra \mathfrak{h}_σ of \mathfrak{h} , by taking complex linear span of σ . Similarly, the subalgebras $\mathfrak{t}_\sigma, \mathfrak{a}_\sigma$ are defined by intersecting \mathfrak{h}_σ with $\mathfrak{t}, \mathfrak{a}$ respectively. These subalgebras define the subgroups $H_\sigma, T_\sigma, A_\sigma$ of H, T, A respectively. A bijective correspondence between the cells $\{\sigma\}$ and the parabolic subgroups $\{P\}$ containing B is given by Langlands decomposition ([10] p.132)

$$(1.2) \quad P = MA_\sigma N_\sigma.$$

Fix a parabolic subgroup P containing B , with σ its corresponding cell. Since H_σ is the normalizer of (P, P) in H , it acts on $G/(P, P)$ on the right. Out of the action of the complex group $G \times H_\sigma$, we shall consider the action of the maximal compact group $K \times T_\sigma$ on $G/(P, P)$. We shall show that

Theorem I *Let ω be a K -invariant Kaehler structure on $G/(P, P)$. Then ω is $K \times T_\sigma$ -invariant if and only if it has a potential function.*

Though we shall be interested mostly in Kaehler structures, Theorem I holds also for degenerate $(1,1)$ -form ω . In the next theorem, we shall derive a necessary and

sufficient condition for a $(1,1)$ -form ω to be Kaehler. Let ω be a $K \times T_\sigma$ -invariant $(1,1)$ -form, so that

$$\omega = \sqrt{-1} \partial \bar{\partial} F,$$

for some function F on $G/(P, P)$. Averaging by the compact group K if necessary, we may assume that F is K -invariant. Let K^σ be the centralizer of T_σ in K . It defines a compact semi-simple subgroup K_{ss}^σ of K , given by $K_{ss}^\sigma = (K^\sigma, K^\sigma)$. We shall show that, as real manifolds and $K \times H_\sigma$ -spaces,

$$(1.3) \quad G/(P, P) = (K/K_{ss}^\sigma) \times A_\sigma.$$

Therefore, the potential function F , being K -invariant, can be regarded as a function on A_σ . Since the exponential map identifies the vector space \mathfrak{a}_σ with A_σ , F becomes a function on \mathfrak{a}_σ . The almost complex structure identifies the dual spaces $\mathfrak{a}_\sigma^* \cong \mathfrak{t}_\sigma^*$, hence the Legendre transform of F can be written as

$$L_F : \mathfrak{a}_\sigma \longrightarrow \mathfrak{t}_\sigma^*.$$

The significance of this map will become apparent shortly, when we study the moment map. We write $\log : A_\sigma \longrightarrow \mathfrak{a}_\sigma$ for the inverse of the exponential map.

The K -action on $G/(P, P)$ preserving ω is Hamiltonian: there exists a unique moment map

$$\Phi : G/(P, P) \longrightarrow \mathfrak{k}^*$$

corresponding to this action. Since Φ is K -equivariant, (1.3) implies that it is determined by its value on $A_\sigma \subset (K/K_{ss}^\sigma) \times A_\sigma$, where A_σ is imbedded as its product with the identity coset $eK_{ss}^\sigma \in K/K_{ss}^\sigma$. Meanwhile, since \mathfrak{k} is semi-simple, the Killing form on \mathfrak{k} is non-degenerate; which induces the inclusion $\mathfrak{t}^* \subset \mathfrak{k}^*$ from $\mathfrak{t} \subset \mathfrak{k}$.

Theorem II *Let ω be a $K \times T_\sigma$ -invariant $(1,1)$ -form on $G/(P, P)$. Then its moment map Φ and its potential function F satisfy $\Phi(a) = \frac{1}{2} L_F(\log a) \in \mathfrak{t}_\sigma^*$ for all $a \in A_\sigma$. Further, $\omega = \sqrt{-1} \partial \bar{\partial} F$ is Kaehler if and only if:*

- (i) $F \in C^\infty(\mathfrak{a}_\sigma)$ is strictly convex; and
- (ii) The image of $\frac{1}{2}L_F$ is contained in the cell $\sigma \subset \mathfrak{t}_\sigma^*$; i.e. $\Phi(A_\sigma) \subset \sigma$.

Since a $K \times T_\sigma$ -invariant Kaehler structure ω has a potential function F , it is exact. Therefore, it is in particular integral. Let \mathbf{L} be the line bundle on $G/(P, P)$ whose Chern class is $[\omega] = 0$, equipped with a connection ∇ whose curvature is ω ([5],[11]). The topology of \mathbf{L} is trivial, but the connection ∇ gives rise to interesting geometry on the holomorphic sections of \mathbf{L} . We recall that \mathbf{L} is equipped with an invariant Hermitian structure \langle, \rangle . Let μ be a $K \times A_\sigma$ -invariant measure on $G/(P, P)$. We consider the integral

$$(1.4) \quad \int_{G/(P, P)} \langle s, s \rangle \mu,$$

for holomorphic sections s of \mathbf{L} . As we shall see in Theorem III, convergence of this integral is determined by the image of the moment map. The $K \times T_\sigma$ -action on $G/(P, P)$ lifts to a $K \times T_\sigma$ -representation on $\mathcal{O}(\mathbf{L})$, the space of holomorphic sections of \mathbf{L} . We similarly define $H_\omega \subset \mathcal{O}(\mathbf{L})$ to be the holomorphic sections in which (1.4) converges. Since μ is K -invariant, H_ω becomes a unitary K -representation space. For a dominant integral weight λ , let $\mathcal{O}(\mathbf{L})_\lambda$ be the holomorphic sections in \mathbf{L} that transform by λ under the right T_σ -action. Since the left K -action commutes with the right T_σ -action, $\mathcal{O}(\mathbf{L})_\lambda$ is a K -representation space. Let σ be the cell corresponding to the parabolic subgroup P , and let $\bar{\sigma}$ be its closure. Then

Theorem III *The irreducible K -representation with highest weight λ occurs in $\mathcal{O}(\mathbf{L})$ if and only if $\lambda \in \bar{\sigma}$. For $\lambda \in \bar{\sigma}$, it occurs with multiplicity one, and is given by $\mathcal{O}(\mathbf{L})_\lambda$. Further, $\mathcal{O}(\mathbf{L})_\lambda$ is contained in H_ω if and only if λ lies in the image of the moment map.*

With this result, it is now clear that in [4], the singular representations are never contained in H_ω :

When $P = B$, σ becomes the open Weyl chamber W . Then Theorem II says that $\Phi(A_\sigma) \subset W$; and by K -equivariance, $\Phi(G/(P, P)) = \text{Ad}_K^*(\Phi(A_\sigma))$ does not intersect the wall $\overline{W} \setminus W$. Consequently, by Theorem III, the irreducible representations $\mathcal{O}(\mathbf{L})_\lambda$ with highest weight $\lambda \in \overline{W} \setminus W$ cannot be contained in H_ω .

Similarly, for general parabolic subgroup P , not all $\mathcal{O}(\mathbf{L})_\lambda$ are contained in H_ω : For $\lambda \in \overline{\sigma} \setminus \sigma$, Theorems II and III say that $\mathcal{O}(\mathbf{L})_\lambda$ exists non-trivially but is not contained in H_ω .

We shall see that, however, for a suitable Kaehler structure ω on $G/(P, P)$, the image of the moment map intersects $\overline{\sigma}$ in all of σ . This way, by Theorem III, all the K -irreducibles $\mathcal{O}(\mathbf{L})_\lambda$ with highest weights $\lambda \in \sigma$ are contained in H_ω . As an application, we provide a geometric construction of a unitary K -representation, containing all the irreducibles with multiplicity one.

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2 KAEHLER STRUCTURES ON $G/(P, P)$

The main purpose of this section is to prove Theorem I. Since K is connected and semi-simple, so is $G = K_{\mathbf{C}}$. Let P be a parabolic subgroup of G containing B , and σ the cell corresponding to P . They are related by Langlands decomposition (1.2)

$$P = MA_{\sigma}N_{\sigma},$$

where A_{σ} is the subgroup described in §1. Then $A_{\sigma} \subset A, N_{\sigma} \subset N$, where A, N come from Iwasawa decomposition of G . Further, A_{σ} normalizes N_{σ} , and is the centralizer of MA_{σ} in A . Therefore, $H_{\sigma} = T_{\sigma}A_{\sigma}$ is the normalizer of $(P, P) = (M, M)N_{\sigma}$ in H , which induces a natural right H_{σ} -action on $G/(P, P)$. We shall give another description of $G/(P, P)$, which reflects this right action better.

Since G is semi-simple, the Killing form is non-degenerate. Let $\mathfrak{a}_{\sigma}^{\perp}$ be the ortho-complement of \mathfrak{a}_{σ} with respect to the Killing form in \mathfrak{a} , and $A_{\sigma}^{\perp} \subset A$ the corresponding subgroup induced by $\mathfrak{a}_{\sigma}^{\perp}$. We construct $\mathfrak{t}_{\sigma}^{\perp}, T_{\sigma}^{\perp}, \mathfrak{h}_{\sigma}^{\perp}, H_{\sigma}^{\perp}$ similarly. Let K^{σ} be the subgroup of K given by

$$K^{\sigma} = \{k \in K ; kt = tk \text{ for all } t \in T_{\sigma}\}.$$

Let $K_{ss}^{\sigma} = (K^{\sigma}, K^{\sigma})$ be the corresponding compact semi-simple Lie group. Then

$$(2.1) \quad (K_{ss}^{\sigma})_{\mathbf{C}} = K_{ss}^{\sigma} A_{\sigma}^{\perp} (M \cap N)$$

is an Iwasawa decomposition of the complexified group $(K_{ss}^{\sigma})_{\mathbf{C}}$. Since $N = (M \cap N) N_{\sigma}$, it follows from (2.1) that

$$\begin{aligned} (2.2) \quad K_{ss}^{\sigma} A_{\sigma}^{\perp} N &= (K_{ss}^{\sigma})_{\mathbf{C}} N_{\sigma} \\ &= (K_{\mathbf{C}}^{\sigma})_{ss} N_{\sigma} \\ &= (MA_{\sigma}, MA_{\sigma}) N_{\sigma} \\ &= (M, M) N_{\sigma} \\ &= (P, P). \end{aligned}$$

Then, the Iwasawa decomposition $G = KAN$ and (2.2) imply that

$$(2.3) \quad G/(P, P) = (K/K_{ss}^\sigma) \times A_\sigma,$$

as real manifolds and $K \times H_\sigma$ -spaces. With this description, the right action of $H_\sigma = T_\sigma A_\sigma$ is clear: T_σ acts on $(K/K_{ss}^\sigma) \times A_\sigma$ simply because it commutes with K_{ss}^σ and A_σ , while A_σ acts on $(K/K_{ss}^\sigma) \times A_\sigma$ by group multiplication on itself. We shall be concerned with the $K \times T_\sigma$ -action on $G/(P, P)$.

Since $N = (B, B) \subset (P, P)$, there is a fibration

$$(2.4) \quad \pi : G/N \longrightarrow G/(P, P).$$

It follows from $G = KAN$ and (2.3) that the fiber of π is $K_{ss}^\sigma \times A_\sigma^\perp$. Further, π sends every right H -orbit in G/N to a right H_σ -orbit in $G/(P, P)$, by contracting each H_σ^\perp -coset to a point.

Given a K -invariant Kaehler structure ω on $G/(P, P)$, we want to show that it is invariant under the right T_σ -action if and only if it has a potential function. Our strategy is to work on the $(1,1)$ -form $\pi^*\omega$ on G/N using results in [4], then transfer this result back to ω . Let V be the orthocomplement of \mathfrak{t} in \mathfrak{k} with respect to the Killing form, so that $\mathfrak{k} = \mathfrak{t} \oplus V$. The Killing form also induces $\mathfrak{t}^* \subset \mathfrak{k}^*$ from $\mathfrak{t} \subset \mathfrak{k}$. If F is a function on A , then by the exponential map, it becomes a function on \mathfrak{a} . Using the almost complex structure, $\mathfrak{a}^* \cong \mathfrak{t}^*$. Therefore, the Legendre transform of F can be written as

$$(2.5) \quad L_F : \mathfrak{a} \longrightarrow \mathfrak{t}^*.$$

Given $\xi \in \mathfrak{k}$, we let ξ^\sharp denote its infinitesimal vector field on G/N induced by the K -action. Let J be the almost complex structure on G/N . For $\eta = J\xi \in \mathfrak{a}$, where $\xi \in \mathfrak{t}$, we define η^\sharp to be the vector field $J\xi^\sharp$. Let $a \in A \subset KA = G/N$. Then its tangent space is $T_a(G/N) = \mathfrak{h}_a^\sharp \oplus V_a^\sharp$. We recall the following result from [4]:

Proposition 2.1 [4] *Let ω be a $K \times T$ -invariant $(1,1)$ -form on G/N . Then*

$\omega = \sqrt{-1}\partial\bar{\partial}F$, where $F \in C^\infty(A)$ by K -invariance. It satisfies $\omega(\mathfrak{h}^\sharp, V^\sharp)_a = 0$. The K -action is Hamiltonian, with moment map $\Phi : G/N \longrightarrow \mathfrak{k}^*$ satisfying

- (i) $\Phi(a) \in \mathfrak{k}^*$ for all $a \in A \subset KA = G/N$;
- (ii) $\Phi : A \longrightarrow \mathfrak{k}^*$ is given by $\Phi(a) = \frac{1}{2}L_F(\log a)$.

Let $m = \dim \sigma, n = \dim \mathfrak{t}$. Let $\{\lambda_1, \dots, \lambda_r\}$ be the positive roots of \mathfrak{g} , where $\{\lambda_1, \dots, \lambda_n\}$ are simple. Here $m \leq n \leq r$. Then $\dim V = 2r$, and $\dim \mathfrak{k} = n + 2r$. In the following proposition, we give a useful decomposition of V . Recall that we define the cell σ in (1.1) using a subset S of the positive simple roots Δ . By switching the roles of S and $\Delta \setminus S$, we can define another cell σ' , with dimension $n - m$. We call σ' the complementary cell to σ . Let J be the almost complex structure on $\mathfrak{k} \oplus \mathfrak{a} = \mathfrak{g}/\mathfrak{n}$. Recall that, V is the orthocomplement of \mathfrak{t} in \mathfrak{k} .

Proposition 2.2 *Let σ, σ' be complementary cells of dimensions $m, n - m$ respectively, where $m \leq n \leq r = \frac{1}{2} \dim V$. There exists a decomposition $V = \bigoplus_1^r V_i$ into two dimensional subspaces V_i . Each V_i is preserved by J and satisfies $[V_i, V_i] \subset \mathfrak{t}$. Further,*

- (i) $\mathfrak{t}_{\sigma'}^\perp = \bigoplus_1^m [V_i, V_i]$,
- (ii) $\mathfrak{t}_\sigma^\perp = \bigoplus_{m+1}^n [V_i, V_i]$.

If ω is a $K \times T$ -invariant $(1,1)$ -form on G/N , then $\omega(V_i^\sharp, V_j^\sharp)_a = 0$ for all $i \neq j, a \in A \subset KA = G/N$.

Proof: Let $\{\lambda_1, \dots, \lambda_r\}$ be the positive roots of \mathfrak{g} , indexed such that the first n of them are simple. Further, we can require that

$$(\lambda_i, \sigma) > 0, (\lambda_i, \sigma') = 0 \ ; \ i = 1, \dots, m,$$

and

$$(\lambda_i, \sigma) = 0, (\lambda_i, \sigma') > 0 \ ; \ i = m + 1, \dots, n,$$

where the pairing taken is the Killing form.

Let $\mathfrak{g}_{\pm i}$ be the root spaces corresponding to $\pm\lambda_i$. Then there exist $\xi_{\pm i} \in \mathfrak{g}_{\pm i}$ such that

$$(2.6) \quad \{ \zeta_i = \xi_i - \xi_{-i} \quad , \quad \gamma_i = \sqrt{-1}(\xi_i + \xi_{-i}) \}_{i=1, \dots, r}$$

form a basis of V ([8] p.421). Here $\{\zeta_i, \gamma_i\}$ are orthogonal to \mathfrak{t} because the root spaces \mathfrak{g}_i are orthogonal to \mathfrak{h} . Further, $\{\xi_{\pm i}\}$ can be chosen such that $[\zeta_i, \gamma_i] \in \mathfrak{t}$, and is dual to $\lambda_i \in \mathfrak{t}^*$ with respect to the Killing form. We define

$$V_i = \mathbf{R}(\zeta_i, \gamma_i).$$

Then $[V_i, V_i] \subset \mathfrak{t}$. Let J be the almost complex structure on $\mathfrak{k} \oplus \mathfrak{a} = \mathfrak{g}/\mathfrak{n}$. From (2.6), it follows that J sends ζ_i to γ_i , and sends γ_i to $-\zeta_i$. Therefore, each V_i is preserved by J .

For $i = 1, \dots, m$, $(\lambda_i, \sigma') = 0$. Since $[\zeta_i, \gamma_i]$ is dual to λ_i , it follows that $[\zeta_i, \gamma_i] \in \mathfrak{t}_{\sigma'}^\perp$. Hence $[V_i, V_i] \subset \mathfrak{t}_{\sigma'}^\perp$ for $i = 1, \dots, m$. But the dual vectors of $\lambda_1, \dots, \lambda_m$ form a basis of $\mathfrak{t}_{\sigma'}^\perp$, hence $\mathfrak{t}_{\sigma'}^\perp = \oplus_1^m [V_i, V_i]$.

For $i = m+1, \dots, n$, $(\lambda_i, \sigma) = 0$. By similar argument, $\mathfrak{t}_\sigma^\perp = \oplus_{m+1}^n [V_i, V_i]$.

Let ω be a $K \times T$ -invariant $(1,1)$ -form on G/N . Suppose that $i \neq j$; we want to show that $\omega(V_i^\sharp, V_j^\sharp)_a = 0$ for $a \in A \subset KA = G/N$. Let $p : \mathfrak{k} \longrightarrow \mathfrak{t}$ be the orthogonal projection, annihilating V . Let $\xi \in V_i, \eta \in V_j$. From (2.6), it follows that $[\xi, \eta]$ is either 0 or in V_k , depending on whether $\lambda_i + \lambda_j$ is some positive root λ_k . In any case,

$$(2.7) \quad p[\xi, \eta] = 0 \quad ; \quad \xi \in V_i, \eta \in V_j.$$

Let $\Phi : G/N \longrightarrow \mathfrak{k}^*$ be the moment map corresponding to the K -action preserving ω . Then $\Phi(a) \in \mathfrak{t}^*$, by Proposition 2.1. Consequently,

$$\begin{aligned} \omega(\xi^\sharp, \eta^\sharp)_a &= (\Phi(a), [\xi, \eta]) \\ &= (\Phi(a), p[\xi, \eta]) \quad \text{since } \Phi(a) \in \mathfrak{t}^* \\ &= 0. \end{aligned}$$

Therefore, $\omega(V_i^\sharp, V_j^\sharp)_a = 0$ for $i \neq j$. This proves the proposition. \square

Let ω be a $K \times T_\sigma$ -invariant Kaehler structure on $G/(P, P)$. Let π be the fibration in (2.4). Then $\pi^*\omega$ is a $K \times TA_\sigma^\perp$ -invariant $(1,1)$ -form on G/N . By Proposition 2.1, it has the form

$$\pi^*\omega = \sqrt{-1}\partial\bar{\partial}f,$$

where f is a K -invariant function on G/N . Since $G/N = KA$, $f \in C^\infty(A)$. We shall show that f can be replaced with another function F which is in the image of

$$\pi^* : C^\infty(G/(P, P)) \longrightarrow C^\infty(G/N),$$

so that we get a potential function for ω .

Let σ be the cell which corresponds to P by (1.2), and σ' its complementary cell. Then σ' defines subgroups $H_{\sigma'}, T_{\sigma'}, A_{\sigma'}$ of H, T, A respectively. By taking the orthocomplements of the Lie algebras $\mathfrak{h}_{\sigma'}, \mathfrak{t}_{\sigma'}, \mathfrak{a}_{\sigma'}$, we construct the subgroups $H_{\sigma'}^\perp, T_{\sigma'}^\perp, A_{\sigma'}^\perp$ as before. Note in particular that $A = A_\sigma^\perp A_{\sigma'}^\perp$. Define $F \in C^\infty(A)$ by

$$(2.8) \quad F = \rho^*f, \quad \rho : A \longrightarrow A_{\sigma'}^\perp \longrightarrow A;$$

where ρ is the composite function of the submersion $A \longrightarrow A_{\sigma'}^\perp$ annihilating A_σ^\perp , followed by the inclusion $A_{\sigma'}^\perp \longrightarrow A$. By $G/N = KA$, F extends uniquely to be a $K \times TA_\sigma^\perp$ -invariant function on G/N . Note that F is in the image of π^* . We define the $K \times TA_\sigma^\perp$ -invariant $(1,1)$ -form

$$\Omega = \sqrt{-1}\partial\bar{\partial}F.$$

We shall show that

$$(2.9) \quad \Omega = \pi^*\omega.$$

Here both Ω and $\pi^*\omega$ are $K \times TA_\sigma^\perp$ -invariant. Since $G/N = KA_\sigma^\perp A_\sigma^\perp$, we only have to compare them at $a \in A_\sigma^\perp$. Also, Proposition 2.1 says that \mathfrak{h}_a^\sharp and V_a^\sharp are complementary with respect to both Ω_a and $\pi^*\omega_a$. Therefore, (2.9) will follow if we can show that

$$(2.10) \quad \Omega(\xi^\sharp, \eta^\sharp)_a = \pi^*\omega(\xi^\sharp, \eta^\sharp)_a \quad ; \quad \xi, \eta \in \mathfrak{h} \text{ or } \xi, \eta \in V, \quad a \in A_\sigma^\perp.$$

This will be checked by the following two lemmas. Recall that $L_F, L_f : \mathfrak{a} \longrightarrow \mathfrak{t}^*$ are the Legendre transforms of F and f , described in (2.5).

Lemma 2.3 $\Omega(\xi^\sharp, \eta^\sharp)_a = \pi^*\omega(\xi^\sharp, \eta^\sharp)_a$ for all $\xi, \eta \in V, a \in A_{\sigma'}^\perp$.

Proof: By Proposition 2.2, the spaces $(V_1)_a^\sharp, \dots, (V_r)_a^\sharp$ are pairwise complementary with respect to Ω_a and $\pi^*\omega_a$, $a \in A_{\sigma'}^\perp$. Therefore, to prove the statement in this lemma, we may consider $\xi, \eta \in V_i$ for each component V_i separately. Since each V_i is two dimensional, it suffices to consider $\xi = \zeta_i, \eta = \gamma_i$. Let

$$\Phi_F, \Phi_f : G/N \longrightarrow \mathfrak{k}^*$$

be the moment maps of the K -actions preserving $\Omega, \pi^*\omega$ respectively. We recall from Proposition 2.1 that $\Phi_F(a) = \frac{1}{2}L_F(\log a), \Phi_f(a) = \frac{1}{2}L_f(\log a)$. We follow the indices $i = 1, \dots, r$ used in Proposition 2.2, as well as the cells σ, σ' of dimensions $m, n - m$ respectively. In what follows, we break up our arguments into three cases, according to the different values of the index i .

Case 1: $i = 1, \dots, m$.

$$\begin{aligned} \Omega(\zeta_i^\sharp, \gamma_i^\sharp)_a &= (\Phi_F(a), [\zeta_i, \gamma_i]) \\ &= (\tfrac{1}{2}L_F(\log a), [\zeta_i, \gamma_i]). \end{aligned}$$

By Proposition 2.2, $[\zeta_i, \gamma_i] \in \mathfrak{t}_{\sigma'}^\perp$, for $i = 1, \dots, m$. By (2.8), $L_F(\log a)$ and $L_f(\log a)$ agree on $\mathfrak{t}_{\sigma'}^\perp$, for $a \in A_{\sigma'}^\perp$. Therefore, the last expression is

$$\begin{aligned} (\tfrac{1}{2}L_f(\log a), [\zeta_i, \gamma_i]) &= (\Phi_f(a), [\zeta_i, \gamma_i]) \\ &= \pi^*\omega(\zeta_i^\sharp, \gamma_i^\sharp)_a. \end{aligned}$$

Case 2: $i = m + 1, \dots, n$.

We recall (2.6), which implies that

$$(2.11) \quad [v, \zeta_i] = \sqrt{-1}(\lambda_i, v)\gamma_i, \quad [v, \gamma_i] = -\sqrt{-1}(\lambda_i, v)\zeta_i$$

for all $v \in \mathfrak{t}$. Therefore, the Lie algebra \mathfrak{k}^σ of K^σ is given by

$$\mathfrak{k}^\sigma = \{\xi \in \mathfrak{k} ; [\xi, \sigma] = 0\} = \mathfrak{t} \oplus_{(\lambda_i, \sigma)=0} V_i.$$

The center of this Lie algebra is \mathfrak{t}_σ , hence the semi-simple Lie algebra \mathfrak{k}_{ss}^σ is given by

$$(2.12) \quad \mathfrak{k}_{ss}^\sigma = \mathfrak{t}_\sigma^\perp \oplus_{(\lambda_i, \sigma)=0} V_i.$$

For $i = m+1, \dots, n$, $(\lambda_i, \sigma) = 0$; hence $\zeta_i, \gamma_i \in \mathfrak{k}_{ss}^\sigma$. But K_{ss}^σ is in the fiber of π , so $\iota(\xi^\sharp)\pi^*\omega_a = 0$ for all $\xi \in V_i$.

We shall show that

$$\iota(\xi^\sharp)\Omega_a = 0$$

for all $\xi \in V_i$. Since each V_i is two dimensional, this will follow if we can show that $\Omega(\zeta_i^\sharp, \gamma_i^\sharp)_a = 0$, for $i = m+1, \dots, n$. But

$$\Omega(\zeta_i^\sharp, \gamma_i^\sharp)_a = \left(\frac{1}{2}L_F(\log a), [\zeta_i, \gamma_i]\right) = 0,$$

since $[\zeta_i, \gamma_i] \in \mathfrak{t}_\sigma^\perp$ and by (2.8), $L_F(\log a)$ vanishes there.

Case 3: $i = n+1, \dots, r$.

From Cases 1, 2, we see that $L_F(\log a), L_f(\log a) \in \mathfrak{t}^*$ agree on the spaces $\mathfrak{t}_\sigma^\perp, \mathfrak{t}_{\sigma'}^\perp$.

Since $\mathfrak{t} = \mathfrak{t}_\sigma^\perp \oplus \mathfrak{t}_{\sigma'}^\perp$, it follows that $L_F(\log a) = L_f(\log a) \in \mathfrak{t}^*$. Therefore,

$$\begin{aligned} \Omega(\zeta_i^\sharp, \gamma_i^\sharp)_a &= (\Phi_F(a), [\zeta_i, \gamma_i]) \\ &= \left(\frac{1}{2}L_F(\log a), [\zeta_i, \gamma_i]\right) \\ &= \left(\frac{1}{2}L_f(\log a), [\zeta_i, \gamma_i]\right) \\ &= (\Phi_f(a), [\zeta_i, \gamma_i]) \\ &= \pi^*\omega(\zeta_i^\sharp, \gamma_i^\sharp)_a. \end{aligned}$$

This proves Lemma 2.3. □

Lemma 2.4 $\Omega(\xi^\sharp, \eta^\sharp)_a = \pi^*\omega(\xi^\sharp, \eta^\sharp)_a$ for all $\xi, \eta \in \mathfrak{h}, a \in A_{\sigma'}^\perp$.

Proof: Let $\mathfrak{h}_\sigma, \mathfrak{h}_{\sigma'}$ denote the subalgebras of \mathfrak{h} , by taking the complex linear spans of σ, σ' respectively. Let $\mathfrak{h}_\sigma^\perp, \mathfrak{h}_{\sigma'}^\perp$ denote their orthocomplements with respect to the Killing form. Then $\mathfrak{h} = \mathfrak{h}_\sigma^\perp \oplus \mathfrak{h}_{\sigma'}^\perp$.

Case 1: $\xi, \eta \in \mathfrak{h}_{\sigma'}^\perp$.

Let $\iota : H_{\sigma'}^\perp \longrightarrow H$ denote the inclusion. From (2.8), we get

$$\sqrt{-1}\partial\bar{\partial}(\iota^*F) = \sqrt{-1}\partial\bar{\partial}(\iota^*f),$$

where $\partial, \bar{\partial}$ are Dolbeault operators on $H_{\sigma'}^\perp$ here. Therefore, given $a \in A_{\sigma'}^\perp \subset H_{\sigma'}^\perp$,

$$\begin{aligned} \Omega(\xi^\sharp, \eta^\sharp)_a &= (\iota^*\Omega)(\xi^\sharp, \eta^\sharp)_a \\ &= (\sqrt{-1}\partial\bar{\partial}(\iota^*F))(\xi^\sharp, \eta^\sharp)_a \\ &= (\sqrt{-1}\partial\bar{\partial}(\iota^*f))(\xi^\sharp, \eta^\sharp)_a \\ &= (\iota^*\pi^*\omega)(\xi^\sharp, \eta^\sharp)_a \\ &= \pi^*\omega(\xi^\sharp, \eta^\sharp)_a. \end{aligned}$$

Case 2: $\xi \in \mathfrak{h}_\sigma^\perp$.

We shall show that

$$(2.13) \quad \iota(\xi^\sharp)\pi^*\omega_a = \iota(\xi^\sharp)\Omega_a = 0,$$

which completes the proof of this lemma. Since $\pi^*\omega$ and Ω are $(1,1)$ -forms, it suffices to check (2.13) for $\xi \in \mathfrak{t}_\sigma^\perp$.

The fiber of π is $K_{ss}^\sigma \times A_\sigma^\perp$, which contains H_σ^\perp . Therefore,

$$\iota(\xi^\sharp)\pi^*\omega_a = 0.$$

We observe that, as complex manifolds,

$$H = \mathbf{C}^n / \mathbf{Z}^n, H_\sigma^\perp = \mathbf{C}^{n-m} / \mathbf{Z}^{n-m}, H_{\sigma'}^\perp = \mathbf{C}^m / \mathbf{Z}^m,$$

and $H = H_\sigma^\perp H_{\sigma'}^\perp$. We introduce complex coordinates $\{z_1, \dots, z_m\}$ on $H_{\sigma'}^\perp$ as well as $\{z_{m+1}, \dots, z_n\}$ on H_σ^\perp ; so that H adopts the product coordinates. Let $z = x + \sqrt{-1}y$, and we let x, y be the coordinates on T, A respectively. From $H = TA, G/N = KA$ and $T \subset K$, we get a natural holomorphic imbedding $\iota : H \longrightarrow G/N$. Then ι^*F , being T -invariant, is a function on y only. For simplicity we still denote it as F . It follows from (2.8) that

$$\frac{\partial F}{\partial y_i} = 0 \text{ for } i = m+1, \dots, n.$$

Therefore, for $a \in A_{\sigma'}^{\perp}$,

$$\begin{aligned}
(2.14) \quad \iota(\xi^{\sharp})(\iota^*\Omega)_a &= \iota(\xi^{\sharp})(\sqrt{-1}\partial\bar{\partial}F)_a \\
&= \iota(\xi^{\sharp})\left(\frac{1}{2}\sum_{j,k=1}^n \frac{\partial^2 F}{\partial y_j \partial y_k} dx_j \wedge dy_k\right) \\
&= \iota(\xi^{\sharp})\left(\frac{1}{2}\sum_{j,k=1}^m \frac{\partial^2 F}{\partial y_j \partial y_k} dx_j \wedge dy_k\right).
\end{aligned}$$

On the other hand, since $\xi \in \mathfrak{t}_{\sigma}^{\perp}$, the vector field ξ^{\sharp} on H is of the form

$$\xi^{\sharp} = \sum_{m+1}^n c_i \frac{\partial}{\partial x_i}.$$

This, together with (2.14), imply that

$$\iota(\xi^{\sharp})\Omega_a = 0.$$

This proves (2.13). Combining the results in Cases 1,2, we have proved Lemma 2.4.

□

Lemmas 2.3 and 2.4 imply (2.10), and hence (2.9). Namely, we have shown that given a $K \times T_{\sigma}$ -invariant Kaehler structure ω on $G/(P, P)$, there exists a function F , which is in the image of π^* by virtue of (2.8), such that

$$\pi^*\omega = \sqrt{-1}\partial\bar{\partial}F.$$

Since F is in the image of π^* , and since π^* is injective, it follows that ω itself has a potential function.

Conversely, suppose that a K -invariant Kaehler structure ω on $G/(P, P)$ has a potential function F . Averaging by the compact group K if necessary, we may assume that F is K -invariant. But by (2.3), this means that F is just a function on A_{σ} , and is automatically $K \times T_{\sigma}$ -invariant. Then ω is also $K \times T_{\sigma}$ -invariant. This proves Theorem I.

We note that our arguments do not require ω to be positive definite. Namely, Theorem I holds even if ω is merely a K -invariant $(1,1)$ -form. In the next section, we use the moment map to derive a necessary and sufficient condition for a $K \times T_{\sigma}$ -invariant $(1,1)$ -form to be Kaehler.

3 MOMENT MAP

Let ω be a $K \times T_\sigma$ -invariant $(1,1)$ -form on $G/(P, P)$, with moment map

$$\Phi : G/(P, P) \longrightarrow \mathfrak{k}^*$$

corresponding to the Hamiltonian action of K on $G/(P, P)$ preserving ω . It is easy to see that this action is Hamiltonian; either from the semi-simplicity of K ([6], §26), or from the fact that $\omega = \sqrt{-1}\partial\bar{\partial}F$ implies $\omega = d\beta$ for some K -invariant real 1-form β ([1], Theorem 4.2.10). We shall study the moment map Φ , and derive a necessary and sufficient condition for ω to be Kaehler.

Suppose now that ω is a $K \times T_\sigma$ -invariant Kaehler structure. We want to derive the two conditions stated in Theorem II. By Theorem I, ω has a potential function F . Averaging by K if necessary, we may assume that F is K -invariant. By (2.3), $G/(P, P) = (K/K_{ss}^\sigma) \times A_\sigma$; so the K -invariant function F is just a function on A_σ . Let π be the fibration in (2.4). Then

$$\Phi \circ \pi : G/N \longrightarrow \mathfrak{k}^*$$

is the moment map corresponding to the K -action on $(G/N, \pi^*\omega)$. Recall that P corresponds to a cell σ via (1.2). Also, $G/N = KA$ and $G/(P, P) = (K/K_{ss}^\sigma) \times A_\sigma$ induce the inclusions

$$A \hookrightarrow \{e\} \times A \subset KA = G/N \quad , \quad A_\sigma \hookrightarrow \{eK_{ss}^\sigma\} \times A_\sigma \subset (K/K_{ss}^\sigma) \times A_\sigma = G/(P, P) \quad .$$

Therefore, we can regard A and A_σ as contained in G/N and $G/(P, P)$ respectively. Note that $\pi(A) = A_\sigma$. From Proposition 2.1, we see that

$$(\Phi \circ \pi)(A) \subset \mathfrak{k}^*.$$

Since the fibration π sends A to A_σ , it follows that $\Phi(A_\sigma) \subset \mathfrak{k}^*$. By K -equivariance of Φ , $\Phi|_{A_\sigma}$ determines Φ entirely. The exponential map from \mathfrak{a}_σ to A_σ is a diffeomorphism, and we let \log be its inverse. This way, the potential function F becomes a

function on \mathfrak{a}_σ . Then, by the almost complex structure, $\mathfrak{a}_\sigma^* \cong \mathfrak{t}_\sigma^*$. Consequently, the Legendre transform of F is

$$L_F : \mathfrak{a}_\sigma \longrightarrow \mathfrak{t}_\sigma^*.$$

We shall show that

$$\Phi : A_\sigma \longrightarrow \mathfrak{t}^*$$

is given by $\Phi(a) = \frac{1}{2}L_F(\log a)$ for all $a \in A_\sigma$. Let

$$\iota : H_\sigma \longrightarrow G/(P, P)$$

be the natural holomorphic imbedding of $H_\sigma = T_\sigma A_\sigma$. Then $\iota^*\omega$ is a T_σ -invariant Kaehler structure on $T_\sigma A_\sigma$, with potential function ι^*F . For simplicity, we still write ι^*F as F . Let m be the dimension of the cell σ . Then, as a complex manifold, $H_\sigma = \mathbf{C}^m/\mathbf{Z}^m$. Therefore, we can introduce complex coordinates $\{z_1, \dots, z_m\}$ on H_σ , where

$$(3.1) \quad \begin{aligned} H_\sigma &= \mathbf{C}^m/\mathbf{Z}^m = \{z_1, \dots, z_m\}, \quad T_\sigma = \mathbf{R}^m/\mathbf{Z}^m = \{x_1, \dots, x_m\}, \\ A_\sigma &= \mathbf{R}^m = \{y_1, \dots, y_m\}, \quad z_i = x_i + \sqrt{-1}y_i. \end{aligned}$$

Since F is T_σ -invariant, it is a function on y only. Then $\iota^*\omega$ becomes (here $\partial, \bar{\partial}$ are Dolbeault operators on H_σ)

$$(3.2) \quad \iota^*\omega = \sqrt{-1}\partial\bar{\partial}F = \frac{1}{2} \sum_{j,k=1}^m \frac{\partial^2 F}{\partial y_j \partial y_k} dx_j \wedge dy_k,$$

where $F \in C^\infty(\mathbf{R}^m)$. Since ω is Kaehler, so is $\iota^*\omega$; and (3.2) says that $\iota^*\omega$ is Kaehler if and only if the Hessian matrix of F is positive definite, i.e. F is strictly convex.

The moment map Φ of the K -action on $(G/(P, P), \omega)$ restricts to be the moment map Φ' of the T_σ -action on $(T_\sigma A_\sigma, \iota^*\omega)$. Let

$$\beta = -\frac{1}{2} \sum_{j=1}^m \frac{\partial F}{\partial y_j} dx_j$$

be a T_σ -invariant 1-form on $T_\sigma A_\sigma$. From (3.2), it follows that $d\beta = \iota^*\omega$, so the moment

map Φ' of the T_σ -action is

$$\begin{aligned}
(\Phi'(ta), \xi) &= -(\beta, \xi^\sharp)(ta) \\
&= (\tfrac{1}{2} \sum_{j=1}^m \frac{\partial F}{\partial y_j} dx_j, \sum_{k=1}^m \xi_k \frac{\partial}{\partial x_k})(ta) \\
&= \tfrac{1}{2} \sum_{j=1}^m \frac{\partial F}{\partial y_j}(a) \xi_j \\
&= \tfrac{1}{2} (L_F(a), \xi),
\end{aligned}$$

where $ta \in T_\sigma A_\sigma, \xi \in \mathfrak{t} = \mathbf{R}^m$. Our computation identifies \mathfrak{a} with A by the exponential map, so in fact $\Phi'(ta) = \tfrac{1}{2} L_F(\log a)$ for all $ta \in T_\sigma A_\sigma$. But Φ and Φ' agree on A_σ , so $\Phi(a) = \tfrac{1}{2} L_F(\log a)$. Hence $\Phi(A_\sigma) \subset \mathfrak{t}_\sigma^*$. We claim further that $\Phi(A_\sigma) \subset \sigma$:

Let $V_i \subset V \subset \mathfrak{k}$ be the subspaces constructed in Proposition 2.2, and let $\{\zeta_i, \gamma_i\} \in V_i$ be the vectors in (2.6). Recall that these indices are made with respect to the positive roots $\{\lambda_i\}$. Since $G/(P, P) = (K/K_{ss}^\sigma) \times A_\sigma$, the infinitesimal vector fields $\zeta_i^\sharp, \gamma_i^\sharp$ on $G/(P, P)$ are non-zero if and only if $\zeta_i, \gamma_i \notin \mathfrak{k}_{ss}^\sigma$. By (2.12), this is equivalent to $(\lambda_i, \sigma) > 0$. Let J be the almost complex structure in $G/(P, P)$, $a \in A_\sigma$, and $(\lambda_i, \sigma) > 0$ so that $\zeta_i^\sharp, \gamma_i^\sharp \neq 0$. By (2.6), $J\zeta_i = \gamma_i$. Since ω is Kaehler,

$$\begin{aligned}
(3.3) \quad 0 &< \omega(\zeta_i^\sharp, J\zeta_i^\sharp)_a \\
&= \omega(\zeta_i^\sharp, \gamma_i^\sharp)_a \\
&= (\Phi(a), [\zeta_i, \gamma_i]) \\
&= (\Phi(a), \lambda_i).
\end{aligned}$$

We have shown that, for all $a \in A_\sigma$, $(\Phi(a), \lambda_i) > 0$ whenever λ_i is a positive root satisfying $(\lambda_i, \sigma) > 0$. This, together with $\Phi(A_\sigma) \subset \mathfrak{t}_\sigma^*$, imply that $\Phi(A_\sigma) \subset \sigma$, as claimed.

We have shown that if ω is Kaehler, then the two conditions stated in Theorem II have to be satisfied. We next show that, conversely, these two conditions are sufficient for ω to be Kaehler.

Recall that the infinitesimal vector field ξ^\sharp on $G/(P, P)$ vanishes if $\xi \in \mathfrak{k}_{ss}^\sigma$. Hence the tangent space at $a \in A_\sigma \subset G/(P, P)$ is spanned by $(\mathfrak{k}_{ss}^{\sigma\perp})_a^\sharp, (\mathfrak{a}_\sigma)_a^\sharp$. Here we define η^\sharp for $\eta = J\xi \in \mathfrak{a}_\sigma$ by $\eta^\sharp = J\xi^\sharp$, where $\xi \in \mathfrak{t}_\sigma$. However, it follows from (2.12) that

$$\mathfrak{k}_{ss}^{\sigma\perp} = \mathfrak{t}_\sigma \oplus_{(\lambda_i, \sigma) > 0} V_i,$$

where V_i is the space described in Proposition 2.2. Here the distinct V_i are orthogonal to one another, due to the orthogonality of the root spaces \mathfrak{g}_i ([8] p.166). Consequently, the tangent space at $a \in A_\sigma \subset G/(P, P)$ is

$$(3.4) \quad T_a(G/(P, P)) = (\mathfrak{h}_\sigma)_a^\# \oplus_{(\lambda_i, \sigma) > 0} (V_i)_a^\#.$$

We claim that $\omega(\mathfrak{h}_\sigma^\#, V_i^\#)_a = \omega(V_i^\#, V_j^\#)_a = 0$, for $i \neq j$:

Since J preserves \mathfrak{h}_σ and V_i , and ω is a $(1,1)$ -form, the first part follows if we can show that $\omega(\mathfrak{t}_\sigma^\#, V_i^\#)_a = 0$. Let $p : \mathfrak{k} \longrightarrow \mathfrak{t}$ be the orthogonal projection, annihilating V . Let $\xi \in \mathfrak{t}_\sigma, \eta \in V_i$. Then $p[\xi, \eta] = 0$, by (2.11). Since $\Phi(a) \in \mathfrak{t}^*$ for $a \in A$,

$$\omega(\xi^\#, \eta^\#)_a = (\Phi(a), [\xi, \eta]) = (\Phi(a), p[\xi, \eta]) = 0.$$

Hence $\omega(\mathfrak{h}_\sigma^\#, V_i^\#)_a = 0$. For $i \neq j$, it follows from (2.7) that $p[V_i, V_j] = 0$. So, by similar argument, $\omega(V_i^\#, V_j^\#)_a = 0$ as claimed.

Therefore, by K -invariance of ω and (3.4), the positive definite condition of ω follows if we can check that

$$(3.5) \quad \omega(\xi^\#, J\xi^\#)_a > 0 \ ; \ \xi \in \mathfrak{h}_\sigma \text{ or } \xi \in V_i, (\lambda_i, \sigma) > 0, a \in A_\sigma.$$

But they follow from the two conditions of Theorem II: Condition (i) of Theorem II implies that the expression in (3.2) is positive definite and hence (3.5) holds for $\xi \in \mathfrak{h}_\sigma$. Condition (ii) of Theorem II implies that $(\Phi(a), \lambda_i) > 0$ whenever $(\lambda_i, \sigma) > 0$, so it follows from (3.3) that (3.5) holds for $\xi \in V_i$. This proves Theorem II.

4 LINE BUNDLE

Fix a $K \times T_\sigma$ -invariant Kaehler structure ω on $G/(P, P)$. By Theorem I, ω has a potential function F . Recall that P determines the subgroup A_σ by (1.2). By K -invariance and (2.3), we can regard F as a function on A_σ . In particular, the expression $\omega = \sqrt{-1}\partial\bar{\partial}F$ also implies that ω is exact. Hence ω is integral, and there exists a complex line bundle \mathbf{L} on $G/(P, P)$ whose Chern class is $[\omega] = 0$, equipped with a connection ∇ whose curvature is ω , as well as an invariant Hermitian structure \langle, \rangle ([5], [11]). The line bundle \mathbf{L} is trivial since $[\omega] = 0$, but the connection ∇ gives rise to interesting geometry. We say that a section s is holomorphic if ∇s annihilates anti-holomorphic vector fields on $G/(P, P)$. We shall show that the $K \times T_\sigma$ -action on $G/(P, P)$ lifts to a $K \times T_\sigma$ -representation on the space of holomorphic sections of \mathbf{L} . To do this, we shall construct a global trivialization of \mathbf{L} . The following topological property of $G/(P, P)$ is useful in this construction:

Lemma 4.1 $H^1(G/(P, P), \mathbf{C}) = 0$.

Proof: By (2.3), $G/(P, P) = (K/K_{ss}^\sigma) \times A_\sigma$. Since A_σ is Euclidean, it suffices to show that $H^1(K/K_{ss}^\sigma, \mathbf{C}) = 0$.

The fibration $K \longrightarrow K/K_{ss}^\sigma$ induces a long exact sequence of homotopy groups,

$$(4.1) \quad \dots \longrightarrow \pi_1(K) \longrightarrow \pi_1(K/K_{ss}^\sigma) \longrightarrow \pi_0(K_{ss}^\sigma) \longrightarrow \dots$$

However, by ([2] p.223),

$$\pi_1(K) \cong \ker(\exp : \mathfrak{t} \rightarrow T)/\mathbf{Z}(\text{roots of } \mathfrak{k}).$$

Therefore, since K is semi-simple, $\pi_1(K)$ is finite. By compactness of K_{ss}^σ , $\pi_0(K_{ss}^\sigma)$ is finite. Hence $\pi_1(K/K_{ss}^\sigma)$, being caught in the middle in (4.1), is also finite. It follows that

$$H^1(K/K_{ss}^\sigma, \mathbf{C}) \cong \text{Hom}(\pi_1(K/K_{ss}^\sigma), \mathbf{C}) = 0,$$

which proves the lemma. □

We return to our pre-quantum line bundle \mathbf{L} on $G/(P, P)$, corresponding to the $K \times T_\sigma$ -invariant Kaehler structure ω . Let β be the 1-form $-\sqrt{-1}\partial F$, so $d\beta = \omega$. We claim that

Proposition 4.2 *There exists a non-vanishing section s_o on \mathbf{L} , with the property*

$$(4.2) \quad \beta = \frac{1}{\sqrt{-1}} \frac{\nabla s_o}{s_o}.$$

This section is unique up to a non-zero constant multiple, and is holomorphic. Up to a non-zero constant,

$$\langle s_o, s_o \rangle = e^{-F}.$$

Proof: Since $[\omega] = 0$, \mathbf{L} is a trivial bundle; so there exists a nowhere zero section s_1 of \mathbf{L} . Let

$$\alpha = \frac{1}{\sqrt{-1}} \frac{\nabla s_1}{s_1}.$$

By the definition of the curvature form on \mathbf{L} , $d\alpha = \omega$; so $d(\beta - \alpha) = 0$. Since $H^1(G/(P, P), \mathbf{C}) = 0$, there exists a complex-valued function f such that $\beta = \alpha + df$. Let $s_o = (\exp \sqrt{-1}f)s_1$. Then

$$\frac{1}{\sqrt{-1}} \frac{\nabla s_o}{s_o} = \frac{1}{\sqrt{-1}} \frac{\nabla s_1}{s_1} + df = \beta.$$

This proves the existence of a holomorphic section s_o satisfying (4.2).

Suppose that s_1 and s_2 are two sections satisfying this formula. Let $h = \frac{s_2}{s_1}$. Then

$$\frac{1}{\sqrt{-1}} \frac{\nabla s_2}{s_2} = \frac{1}{\sqrt{-1}} \frac{\nabla s_1}{s_1} + \frac{1}{\sqrt{-1}} d \log h,$$

which implies that h is a constant. Hence, up to a constant, the solution of (4.2) is unique.

If v is an anti-holomorphic vector field, then

$$\frac{1}{\sqrt{-1}} \frac{\nabla_v s_o}{s_o} = \iota(v)\beta = 0,$$

as β is a form of type $(1,0)$. Hence s_o is holomorphic. Since β is $K \times T_\sigma$ -invariant, s_o induces a $K \times T_\sigma$ -representation on the space of holomorphic sections on \mathbf{L} , where s_o is $K \times T_\sigma$ -invariant. Namely, given a holomorphic section $f s_o$ of \mathbf{L} (note s_o is non-vanishing), $K \times T_\sigma$ acts by

$$(4.3) \quad L_k^* R_t^* (f s_o) = (L_k^* R_t^* f) s_o ; \quad k \in K, t \in T_\sigma,$$

where $L_k^* R_t^* f$ denotes the standard action on the holomorphic functions lifted from the $K \times T_\sigma$ -action on $G/(P, P)$. Hence s_o defines a $K \times T_\sigma$ -equivariant trivialization.

For this section s_o , we now show that $\langle s_o, s_o \rangle = e^{-F}$. By K -invariance, it suffices to show that this is the case when restricted to A_σ . Let σ be the cell corresponding to the parabolic subgroup P , and let m be the dimension of σ . We write $T_\sigma A_\sigma = \mathbf{C}^m / \mathbf{Z}^m = \{z_1, \dots, z_m\}$ as in (3.1), so that F , being T_σ -invariant, is a function on y only. Let $\iota : T_\sigma A_\sigma \longrightarrow G/(P, P)$ be the natural inclusion. Then

$$(4.4) \quad \iota^* \beta = -\sqrt{-1} \partial F = \frac{1}{2} \sum_1^m \frac{\partial F}{\partial y_i} dz_i.$$

Let $\nabla_i = \nabla_{\frac{\partial}{\partial y_i}}$. Then

$$\frac{\partial}{\partial y_i} \langle s_o, s_o \rangle = \langle \nabla_i s_o, s_o \rangle + \langle s_o, \nabla_i s_o \rangle.$$

However, by (4.2) and (4.4),

$$\frac{\nabla_i s_o}{s_o} = \sqrt{-1} (\beta, \frac{\partial}{\partial y_i}) = -\frac{1}{2} \frac{\partial F}{\partial y_i}$$

so

$$\frac{\partial}{\partial y_i} \log \langle s_o, s_o \rangle = -\frac{\partial F}{\partial y_i}.$$

Therefore, up to a non-zero constant multiple,

$$\langle s_o, s_o \rangle = e^{-F}.$$

This proves the proposition. □

Let $\mathcal{O}(\mathbf{L})$ denote the space of holomorphic sections of the line bundle \mathbf{L} on $G/(P, P)$. By Proposition 4.2, s_o induces a $K \times T_\sigma$ -representation on $\mathcal{O}(\mathbf{L})$, given by (4.3), where s_o is $K \times T_\sigma$ -invariant. Let $\lambda \in \mathfrak{t}_\sigma^*$ be a dominant integral weight, and let $\mathcal{O}(\mathbf{L})_\lambda$ denote the holomorphic sections that transform by λ under the right T_σ -action. Since this action commutes with the left K action, $\mathcal{O}(\mathbf{L})_\lambda$ is a K -subrepresentation of $\mathcal{O}(\mathbf{L})$. We now show that the K -finite vectors in $\mathcal{O}(\mathbf{L})$ decompose into $\{\mathcal{O}(\mathbf{L})_\lambda ; \lambda \in \bar{\sigma}\}$ as irreducible K -representations with highest weights λ . Using the holomorphic section s_o of Proposition 4.2, it suffices to consider the holomorphic functions $\mathcal{O}(G/(P, P))$; since

$$\mathcal{O}(G/(P, P)) \otimes s_o = \mathcal{O}(\mathbf{L})$$

is a $K \times T_\sigma$ -equivariant trivialization.

Recall that \bar{W} is the closure of the Weyl chamber W , and $\sigma \subset \bar{W}$ is the cell corresponding to P . Let $\bar{\sigma}$ denote its closure in \bar{W} . For a dominant integral weight $\lambda \in \mathfrak{t}^*$, let $\mathcal{O}_\lambda \subset \mathcal{O}(G/(P, P))$ denote the holomorphic functions that transform by λ under the right T_σ -action. Since the right T_σ -action commutes with the left K -action, each \mathcal{O}_λ is a K -representation space.

Proposition 4.3 *The irreducible K -representation with highest weight λ occurs in $\mathcal{O}(G/(P, P))$ if and only if $\lambda \in \bar{\sigma}$. For $\lambda \in \bar{\sigma}$, it occurs with multiplicity one, and is given by \mathcal{O}_λ .*

Proof: The fibration π of (2.4) induces an injection of holomorphic functions,

$$\pi^* : \mathcal{O}(G/(P, P)) \longrightarrow \mathcal{O}(G/N).$$

This map intertwines with the $K \times T_\sigma$ -action.

Let λ be a dominant integral weight, but suppose that $\lambda \notin \bar{\sigma}$. We shall show that the K -irreducible with highest weight λ does not occur in $\mathcal{O}(G/(P, P))$. By the Borel-Weil theorem, the K -irreducible with highest weight λ occurs in $\mathcal{O}(G/N)$ with multiplicity one, and can be taken as the holomorphic functions in G/N that

transform by λ under the right T -action. We denote this space by $V_\lambda \subset \mathcal{O}(G/N)$. Since π^* is injective, it suffices to show that

$$(4.5) \quad \pi^* \mathcal{O}(G/(P, P)) \cap V_\lambda = 0.$$

Since $\lambda \notin \bar{\sigma}$, $(\lambda, \xi) \neq 0$ for some $\xi \in \mathfrak{t}_\sigma^\perp$. Let $0 \neq f \in V_\lambda$. Then the right action R_ξ^* on V_λ satisfies

$$(4.6) \quad R_\xi^* f = (\lambda, \xi) f \neq 0.$$

Since T_σ^\perp is in the fiber of π , the image of π^* is T_σ^\perp -invariant. Therefore, (4.6) says that f cannot be in the image of π^* . This proves (4.5).

Conversely, suppose that $\lambda \in \bar{\sigma}$ is a dominant integral weight. We again let $V_\lambda \subset \mathcal{O}(G/N)$ be the holomorphic functions that transform by λ . By the Borel-Weil theorem, V_λ is an irreducible representation with highest weight λ , and such irreducible occurs with multiplicity one. Therefore, to complete the proof of Proposition 4.3, we need to show

$$(4.7) \quad V_\lambda \subset \pi^* \mathcal{O}(G/(P, P)), \quad \lambda \in \bar{\sigma}.$$

Recall from (2.2) that the fiber of π is $(K_{ss}^\sigma)_\mathbb{C}/(M \cap N) = K_{ss}^\sigma \times A_\sigma^\perp$. Choose a fiber of π , and let

$$\iota : K_{ss}^\sigma A_\sigma^\perp \hookrightarrow G/N$$

be a holomorphic $K_{ss}^\sigma \times A_\sigma^\perp$ -equivariant imbedding as this fiber of π . Let $f \in V_\lambda$. We claim that f is constant on this fiber:

By applying the Borel-Weil theorem on $(K_{ss}^\sigma)_\mathbb{C}/(M \cap N) = K_{ss}^\sigma \times A_\sigma^\perp$, we see that $\iota^* f$, which is right T_σ^\perp -invariant since $(\lambda, \mathfrak{t}_\sigma^\perp) = 0$, has to be a constant function. Hence f is constant on that fiber, as claimed.

Since our argument is independent of the choice of fiber and element of V_λ , we conclude that every element of V_λ is constant on every fiber of π . This implies (4.7), and Proposition 4.3 is now proved. \square

We have shown that the irreducible K -representation with highest weight λ occurs in $\mathcal{O}(\mathbf{L})$ if and only if $\lambda \in \bar{\sigma}$. For $\lambda \in \bar{\sigma}$, it occurs with multiplicity one, and is given by $\mathcal{O}(\mathbf{L})_\lambda$. We shall decide which of these irreducible K -representations are square-integrable, in the following sense.

From the description $G/(P, P) = (K/K_{ss}^\sigma) \times A_\sigma$, we see that there is a $K \times A_\sigma$ -invariant measure μ on $G/(P, P)$, which is unique up to non-zero constant. Given a holomorphic section s of \mathbf{L} , we consider the integral

$$\int_{G/(P, P)} \langle s, s \rangle \mu .$$

Let $H_\omega \subset \mathcal{O}(\mathbf{L})$ be the holomorphic sections in which this integral converges. Since the Hermitian structure \langle, \rangle and μ are K -invariant, H_ω becomes a unitary K -representation space. The next proposition shows which irreducible K -representations occur in H_ω .

Let $\lambda \in \bar{\sigma}$ be a dominant integral weight. Let

$$\Phi : G/(P, P) \longrightarrow \mathfrak{k}^*$$

be the moment map of the K -action on $(G/(P, P), \omega)$. Recall that \mathcal{O}_λ and $\mathcal{O}(\mathbf{L})_\lambda$ are respectively the holomorphic functions and sections that transform by $\lambda \in \mathfrak{k}_\sigma^*$ under the right T_σ -action.

Proposition 4.4 *Let $s \in \mathcal{O}(\mathbf{L})_\lambda$. Then $s \in H_\omega$ if and only if λ is in the image of the moment map.*

Proof: Let s_o be the unique holomorphic section of Proposition 4.2. Therefore, $\langle s_o, s_o \rangle = e^{-F}$, where F is the potential function of ω . Since s_o is non-vanishing and $K \times T_\sigma$ -invariant,

$$\mathcal{O}(\mathbf{L})_\lambda = \mathcal{O}_\lambda \otimes s_o.$$

Therefore, we are reduced to showing that $f \in \mathcal{O}_\lambda$ satisfies

$$(4.8) \quad \int_{G/(P, P)} |f|^2 e^{-F} \mu < \infty$$

if and only if λ is in the image of Φ .

Here μ is the product of a K -invariant measure dk on K/K_{ss}^σ and a Haar measure da on A_σ . By the exponential map, the measure da on A_σ can be identified with the Lebesgue measure dy on \mathbf{R}^m , where $m = \dim \sigma$. Given $k \in K$, the left K -action on \mathcal{O}_λ , $L_k^* : \mathcal{O}_\lambda \longrightarrow \mathcal{O}_\lambda$, is

$$(L_k^* f)(p) = f(kp).$$

Let f_1, \dots, f_N be a basis of \mathcal{O}_λ which is orthonormal with respect to the (unique) K -invariant inner product on \mathcal{O}_λ . Given an element $f = \sum c_i f_i$ of \mathcal{O}_λ ,

$$f(ky) = (L_k^* f)(y) = \sum c_i a_{ir}(k) f_r(y),$$

where $a_{ir}(k)$ is the ir th matrix coefficient of the K -representation on \mathcal{O}_λ with respect to the basis above. Thus

$$\int |f(ky)|^2 dk = \sum c_i \overline{c_j} \left(\int a_{ir}(k) \overline{a_{js}(k)} dk \right) f_r(y) \overline{f_s(y)},$$

where the integrals are taken over K/K_{ss}^σ . However, by Peter-Weyl the inner integral is equal to

$$\frac{1}{N} \delta_{ij} \delta_{rs},$$

([3] p.186) so the integral (4.8) reduces to

$$(4.9) \quad \frac{1}{N} \|f\|^2 \int_{\mathbf{R}^m} \sum |f_r(y)|^2 e^{-F(y)} dy,$$

where $\|f\|$ is the norm of f with respect to the given K -invariant inner product structure on \mathcal{O}_λ . However, each of the functions $f_r(y)$ transforms under the infinitesimal \mathfrak{t}_σ -action according to the character $\lambda \in \mathfrak{t}_\sigma^*$, and therefore, being holomorphic, transforms under the action of $\mathfrak{h}_\sigma = (\mathfrak{t}_\sigma)_\mathbf{C}$ according to the complexified character $\lambda_\mathbf{C} \in \mathfrak{h}_\sigma^*$. In particular, $|f_r(y)|^2$ is a constant multiple of $e^{2\lambda(y)}$. Hence if $f \neq 0$, (4.9) is a constant multiple of the integral

$$\int_{\mathbf{R}^m} e^{-F(y) + 2\lambda(y)} dy.$$

However, this integral converges if and only if 2λ is in the image of the Legendre transform of F ([4], Appendix); or equivalently if and only if λ is in the image of the moment map. This proves the proposition. \square

With this result, Theorem III follows. We see from Theorems II, III that not all irreducibles are contained in H_ω : The irreducible representation $\mathcal{O}(\mathbf{L})_\lambda$ with highest weight λ satisfies $\mathcal{O}(\mathbf{L})_\lambda \subset H_\omega$ if and only if $\lambda \in \frac{1}{2}L_F(\mathfrak{a}_\sigma) \subset \sigma$. This necessarily excludes $\lambda \in \bar{\sigma} \setminus \sigma$. However, in the next section, we shall see that the potential function F can be constructed such that $\frac{1}{2}L_F(\mathfrak{a}_\sigma) = \sigma$, and hence $\mathcal{O}(\mathbf{L})_\lambda \subset H_\omega$ for all $\lambda \in \sigma$.

5 CONSTRUCTION OF A MODEL

Let P be a parabolic subgroup of G , and let σ its corresponding cell of dimension m , given in (1.2). There exist dominant fundamental weights $\alpha_1, \dots, \alpha_m \in \mathfrak{a}_\sigma^*$ ([9] p.498) such that

$$\sigma = \left\{ \sum_1^m y_i \alpha_i ; y_i > 0 \right\}.$$

Let $F_P : \mathfrak{a}_\sigma \longrightarrow \mathbf{R}$ be defined by

$$(5.1) \quad F_P(v) = \sum_1^m e^{\alpha_i(v)}.$$

Then $F_P \in C^\infty(\mathfrak{a}_\sigma)$ is strictly convex, and the image of its Legendre transform is exactly σ . Therefore, the moment map Φ satisfies $\Phi(A_\sigma) = \sigma$. Extend F_P to $G/(P, P)$ by K -invariance, and it follows from Theorem II that

$$\omega_P = \sqrt{-1} \partial \bar{\partial} F_P$$

is a Kaehler structure on $G/(P, P)$. Let \mathbf{L}_P be the corresponding line bundle, described before. For a dominant integral weight λ , we let $\mathcal{O}(\mathbf{L}_P)_\lambda$ denote the holomorphic sections of \mathbf{L}_P that transform by λ under the right T_σ -action. Let H_{ω_P} be the holomorphic sections that are square-integrable under (1.4), so that it is a unitary K -representation space. By Theorem III, $\mathcal{O}(\mathbf{L}_P)_\lambda$ is an irreducible K -representation with highest weight λ , whenever $\lambda \in \bar{\sigma}$. Further, since $\Phi(A_\sigma) = \sigma$, $\mathcal{O}(\mathbf{L}_P)_\lambda \subset H_{\omega_P}$ whenever $\lambda \in \sigma$.

We repeat this geometric construction among all the parabolic subgroups P containing the fixed Borel subgroup $B = HN$. In each case, we use F_P in (5.1) as the potential function for the Kaehler structure ω_P on $G/(P, P)$. Then the direct sum

$$\oplus_{B \subset P} H_{\omega_P}$$

is a model in the sense of I.M. Gelfand [7]: a unitary K -representation where all irreducibles occur with multiplicity one.

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